Table 1 Convergence history

Iteration number	Value of cost functional J	$\int_{t_0}^{t_f} \tilde{H}_u \tilde{H}_u^T \, dt$	$u(0)$	$u(0.5)$	$u(1)$
0	−4.07436	4.1196	$\pi/2$	$\pi/4$	0
0_ψ	−3.29834	9.5861	1.571	0.946	−0.932
1	−3.49072	0.7171	1.295	1.092	−1.032
2	−3.50696	0.0312	1.383	1.053	−0.994
3	−3.50781	0.0077	1.357	1.042	−0.984
4	−3.50800	0.0022	1.374	1.038	−0.979
5	−3.50806	0.0008	1.363	1.036	−0.978
Optimal[7]	−3.50809	0	1.367	1.034	−0.976

interval divided into 100 uniform segments. A quadratic polynomial approximation scheme was used for each one-dimensional minimization.

As a numerical example, consider the rocket problem treated in Ref. 7. In normalized form the 3rd-order dynamical system is

$$\dot{x}_1 = x_2, \quad x_1(0) = 0, \quad x_1(1) = 1.0$$
$$\dot{x}_2 = 6.4 \sin u - 3.2, \quad x_2(0) = 0, \quad x_2(1) = 0$$
$$\dot{x}_3 = 6.4 \cos u, \quad x_3(0) = 0$$

and the cost functional is $J = -x_3(1)$.

The convergence history obtained by using the proposed algorithm is given in Table 1. The iteration number 0_ψ refers to the control $[u_0(t)]_\psi$, where the initial control $u_0(t) = (\pi/2)(1-t)$. Since the differential equations of this problem are linear in the state variables, the transition matrix was calculated analytically, thus eliminating any numerical integration in the calculation[2] of the projection operator. The total execution time required for this solution was 19.2 sec. Figure 1 contains plots of $J[u_{i-1}+\theta_{i-1}p_{i-1}]$ and $J[(u_{i-1}+\theta_{i-1}p_{i-1})_\psi]$ vs the stepsize parameter θ_{i-1} for $i = 1, 5$. The $J(u)$ curves have lower minima since the constraints are violated. It should be noted that the minimum along the $J(u_\psi)$ curve is not in a region where the linearized dynamics are valid, thus lending support to the stepsize selection policy proposed by Willoughby.[6]

Conclusions

These numerical results demonstrate the feasibility of using this technique. The fact that this method is not dependent upon guessing additional algorithm parameters will hopefully make it a powerful tool in the future. The idea of using a one-dimensional minimization will be of greater importance when a conjugate gradient or Davidon algorithm is used since the generation of conjugate directions is highly dependent upon an accurate one-dimensional minimization. In light of the encouraging results resulting from using steepest descent on a rather simple problem, further investigation using a more advanced algorithm on a more sophisticated problem is warranted.

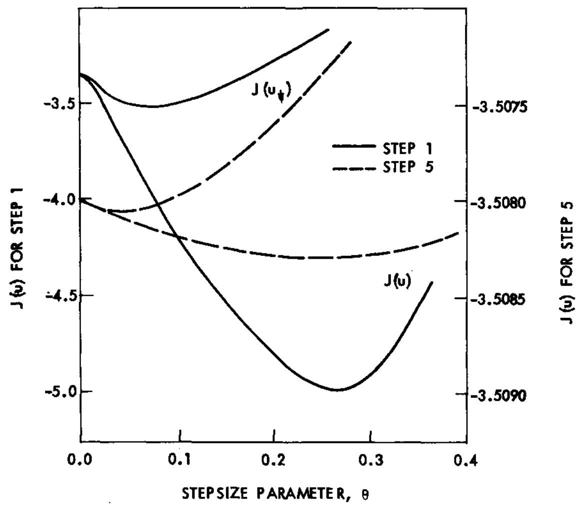

Fig. 1 One-dimensional search functional value profile.

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Solution of a Plastic Buckling Paradox

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1. Introduction

It is well known that the available experimental results on plastic buckling of plates and shells do not agree with a J_2-type flow theory but that they agree very well with a deformation-type theory or to a certain degree with a slip theory.[1,2] Taking into account that the deformation type theory as a class has been discredited because of inconsistencies[3] and that the slip theory has not been improved to an extent that it will satisfy experimental results,[4] it is worthwhile to show some new experimental results with yield surfaces because of their implications for solving this long standing problem of understanding plastic buckling.

The basis of the misunderstanding concerning plastic buckling may lie in the fact that it has not been recognized well experimentally that materials become anisotropic practically immediately upon plastic loading so that the use of an isotropic flow theory of plasticity is not appropriate particularly for the solution of problems in which the appearance of a very small plastic strain is of great importance; this is, of course, the case in plastic buckling.

2. New Experimental Results

Figures 1 and 2 show typical yield surfaces in a tension-torsion stress space as obtained experimentally for commercially pure aluminum. We shall assume that the same general results are valid for other metals and for other stress-space quadrants such as the compression-torsion space. Indeed initial isotropy requires that

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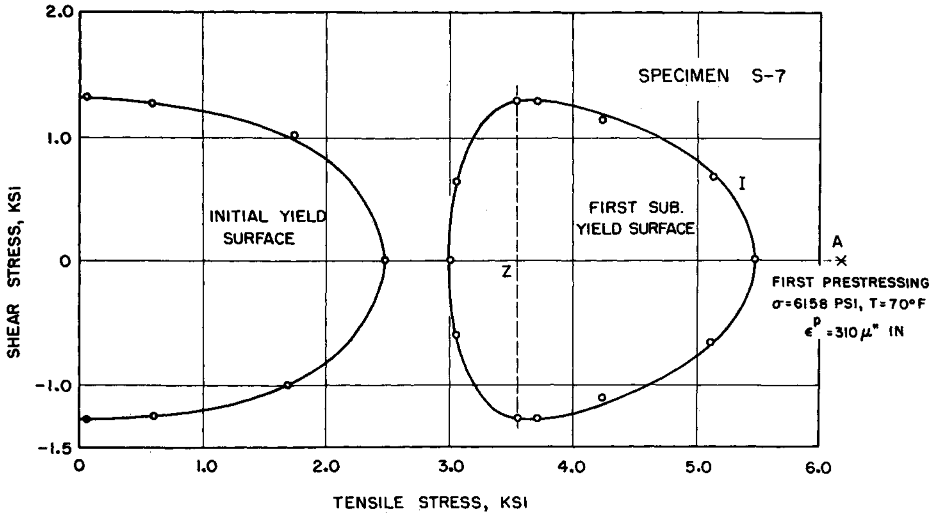

Fig. 1 Yield surface I in a tension-torsion stress space.

the behavior in the compression-torsion space should be the same as in tension-torsion space.

Yield surface I, Fig. 1, corresponds to loading from 0 to the point A while yield surface II, Fig. 2, corresponds to loading to point B from inside the yield surface I after reaching point A and then retreating inside surface I to Z. It is seen that the loading point is not on the yield surface but that the yield surface is trailing behind the loading point. Figures 1 and 2 represent our new experimentally discovered hardening law.[5] The yield surface corresponding to A does not seem to have a corner and it represents anisotropic behavior of the material. We may assume that the loading surfaces† at A and B are parallel to the corresponding yield surfaces I and II. These two yield surfaces I and II have been produced with very little plastic strain, surface I with 210 μin. per inch tensile plastic prestrain at A, and surface II with 200 μin. per inch torsional plastic prestrain at B (in addition to the 210 μin. per inch tensile plastic prestrain already accumulated). We also know experimentally that if the loading path moves from A to any other point inside I and then in a line parallel to ZB in the same direction as ZB then the same hardening law applies and that if the loading path moves from A to C in torsion the same law applies and there will be a trailing of the yield surface in translation behind the loading point accompanied by a deformation in the direction of AC. Such a translation and deformation should occur very soon upon a very small amount of plastic strain of the order of less than a hundred μin. per inch. Yield surface II and the corresponding loading surface should have substantially the same form whether we proceed in the path AZB or in the path AC.

3. Implications of our Experimental Results for Buckling

The comparison between experimental results in plastic buckling of plates and shells and theories of plastic buckling is based at least on the following data: 1) the amount of plastic strain accumulated when buckling occurs; 2) the slope of the stress-strain curve at that point; and 3) the directions of the plastic strain increments predicted by the theories.

Experiments by Pride and Heimerl[6] in plastic buckling of simply supported compressed plates show that, as in Fig. 3, the test results are for plastic strains of a magnitude which is substantial, one thousand to several thousand μin. per inch plastic strain. At that level the slope of the stress-strain curve does not change appreciably with the amount of plastic strain and hence with the theory used. The existing experimental results on buckling are for comparatively substantial plastic strains the magnitude of which, compared to the simultaneously existing elastic strains is of the order of 20% or higher. At this stage, according to our experimental results the yield surface is already highly deformed and it has no resemblance with the one used in the J_2-theory of plastic flow.

The results of the theories depend to a great extent on the ratio $d\gamma''/d\varepsilon''$. This ratio determines to a large extent how fast plastic shearing strains appear as compared to the plastic normal strains. Here, not only the actual value of $d\gamma''/d\varepsilon''$ is important but also the development of $d\gamma''/d\varepsilon''$ as plastic shearing strains accumulate. This development is much faster in the case of an anisotropic yield surface than in the case of an isotropic yield surface. The development of plastic shearing strains determines to a great extent the appearance of plastic buckling and particularly the experimental results.

It is probable that by considering the correct yield surface we shall obtain a better agreement between the plastic flow theory and experimental results. In addition initial imperfections have the tendency to introduce some initial shearing strains and this

† For the definition of a loading surface distinct from the yield surface see Refs. 13, 14.

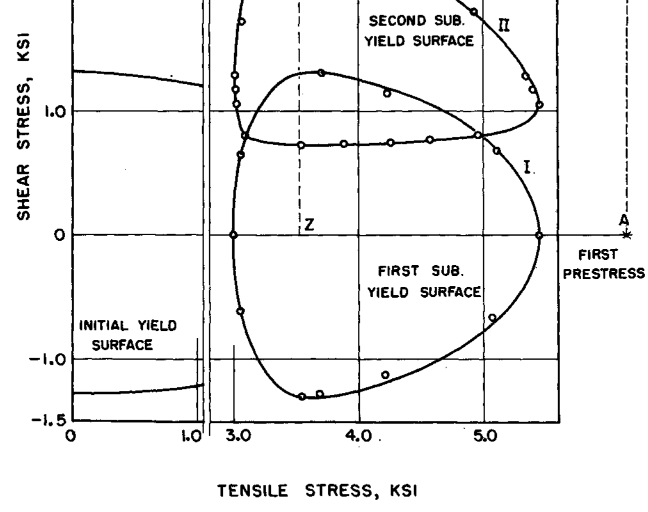

Fig. 2 Yield surface II in a tension-torsion stress space.

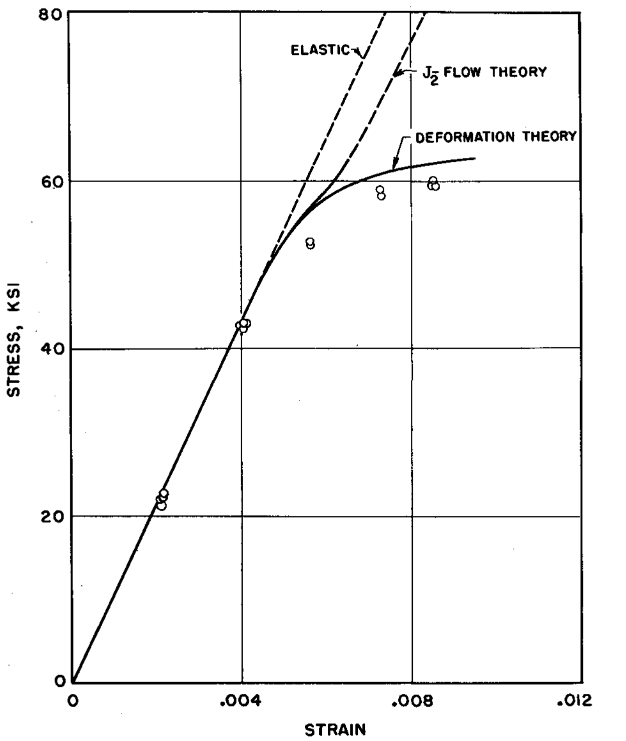

Fig. 3 The test results of experiments in plastic buckling using plastic strains of substantial magnitudes.

produces an accentuating influence in the development of the anisotropic yield surface. In fact we believe that for a smaller initial imperfection an anisotropic yield surface will tend to produce the same results as a large initial imperfection will tend to give with the J_2-theory, which may also explain some questions raised in the literature.[7–9] It is not important whether the initial yield surface has or has not a corner at the σ-axis. It is important how fast the slope $d\gamma''/d\varepsilon''$ changes with the development of the shearing plastic strains. This change is extremely fast, much more than is expected when the J_2-theory is applied.

Additional experiments with present day instrumentation as well as theoretical work is necessary to finally verify these thoughts. For example, experiments on plastic buckling could be performed with the same material used for obtaining the yield surfaces, and then theoretical calculations could be made with the exactly obtained results. Such experiments and theoretical work are now in progress.

3. Final Conclusions

It is seen that a very small plastic strain of the order of a few hundred μin. per inch produces a severe change in the yield surface, that the material becomes anisotropic very fast and the yield surface should be the one of anisotropic flow theory where the anisotropy is due to the motion and deformaton of the yield surface. We believe also that when real yield surfaces are introduced into the picture the problem of plastic buckling may be better understood. Indeed, as Hutchinson and Koiter[10] point out in a recent survey "it is somewhat doubtful that the simplest J_2-theory would be adequate for buckling problems." Additional proper experimental and appropriate theoretical work is necessary.

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Equation for Nonlinear Vibrations of Shells

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THE nonlinear differential equation

$$\ddot{x}+\varepsilon x(x\ddot{x}+\dot{x}^2)+x=0, \qquad \varepsilon>0, \quad 0\leqq t<\infty \tag{1}$$

arises[1] in the theory of nonlinear inextensional vibrations of an infinitely long cylindrical shell. The purpose of this Note is to study the phase plane behavior of the solutions of Eq. (1) and in particular to obtain the period of the motion.

If we write

$$\dot{x}=U \tag{2}$$

then we can write Eq. (1) in the form

$$U=-x[(1+\varepsilon U^2)/(1+\varepsilon x^2)] \tag{3}$$

The differential equation for the solution in the (x, U) phase plane is

$$dU/dx=\dot{U}/\dot{x}=-(x/U)[(1+\varepsilon U^2)/(1+\varepsilon x^2)] \tag{4}$$

which can be integrated to give the one parameter family of phase-paths

$$(1+\varepsilon x^2)(1+\varepsilon U^2)=C \tag{5}$$

where C is a constant of integration to be determined from the initial conditions of the problem by

$$C=\{1+\varepsilon[x(0)]^2\}\{1+\varepsilon[\dot{x}(0)]^2\} \tag{6}$$

These phase-paths are a family of closed curves which are symmetric with respect to x and U and represent periodic motions.

From Eq. (4) we see that the origin of the phase plane, which

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